Flop transitions in M-theory cosmology

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We study flop transitions for M-theory on Calabi-Yau threefolds and their applications to cosmology in the context of the effective five-dimensional supergravity theory. In particular, the additional hypermultiplet which becomes massless at the transition is included in the effective action. We find the potential for this hypermultiplet which includes quadratic and quartic terms as well as additional dependence on the Kähler moduli. By constructing explicit cosmological solutions, it is demonstrated that a flop transition can indeed be achieved dynamically, as long as the hypermultiplet is set to zero. For an explicit example we find that once excitations of the hypermultiplet are taken into account, the transition is generically not completed but the system is classically stabilized close to the transition region, at least as long as the transition states do not vanish. Regions of moduli space close to flop transitions can, therefore, be viewed as preferred by the cosmological evolution.

I. INTRODUCTION

Spacelike curvature singularities arising in cosmological solutions to low-energy string effective actions and their potential resolution constitute a challenging problem in string and M-theory. On the other hand, the string resolution of certain timelike singularities, such as those arising from collapsed cycles in the internal manifold, is, at least in principle, understood. In the course of a string or M-theory phase transition, triggered by cosmological evolution of moduli fields, these singularities may, in fact, arise at a particular instance in time. For example, a flop transition [1,2,5] corresponds to a collapsing two-cycle in the internal Calabi-Yau space while a conifold transition [3–5] corresponds to a collapsing three-cycle. Clearly, such transitions are of interest for string and M-theory early universe cosmology. For example, one would like to know whether the topological transition can actually be realized dynamically, that is, whether the topology of the internal manifold could have indeed changed during the cosmological evolution. Further, one would like to understand, in this cosmological context, the role of the states which become light at the transition and how precisely the transition effects the evolution of the fields.

In this paper we will be answering these questions for the mildest form of topology change, namely the flop. A related discussion, but in the context of black-hole solutions, has been carried out in Refs. [6,7]. We will be working in the context of M-theory on Calabi-Yau threefolds, leading to an effective description in terms of five-dimensional $N = 1$ supergravity theories [8–12]. Flop transitions arise from collapsing two-cycles within the Calabi-Yau manifold and are, therefore, controlled by the Kähler moduli which, together with $U(1)$ gauge fields, are contained in five-dimensional

vector multiplets. Membranes wrapping Calabi-Yau two-cycles lead to hypermultiplet states in five dimensions with a mass proportional to the volume of the cycle. When the cycle collapses at the transition the hypermultiplet becomes massless and can no longer be ignored in the effective theory. In our analysis, we will include these hypermultiplet states explicitly into the five-dimensional effective action. In the following, we will also refer to these states as "transition states." In the case of M-theory on Calabi-Yau threefolds, there are no non-geometrical phases [2], that is, transitions are sharp. This implies that, going through the transition by first collapsing the cycle and then enlarging it up in a topologically different way, leads to a another, topologically distinct Calabi-Yau space. While the Hodge numbers of the original and the "flopped" Calabi-Yau space are the same other topological quantities, such as the intersection numbers, change across the transition.

In terms of the five-dimensional effective supergravity theory, the transition can be described in a non-singular way once the additional hypermultiplet is included. For example, the jump in the intersection numbers which appear in the five-dimensional Chern-Simons term is accounted for by loop corrections involving the hypermultiplet states [2] while the Kähler moduli space metric is continuous across the transition [6]. It turns out that the additional hypermultiplet is charged with respect to a particular linear combination of the vector multiplet gauge fields. Supersymmetry then implies the existence of a potential which depends on the transition states and the vector multiplet scalars. It is this potential which will play an important role in our cosmological analysis. Practically, we will, therefore, study time evolution in Kähler moduli space close to the flop region including the effect of the transition states and their potential.

The plan of the paper is as follows. In Sec. II we will review $N = 1$ supergravity in eleven and five dimensions and the five-dimensional effective action for M-theory on Calabi-Yau threefolds. With this machinery at hand, we then go on

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to derive the effective five-dimensional action of the transition states. Section III analyzes the cosmology of the five-dimensional theory for arbitrary Calabi-Yau spaces, first with vanishing and then with non-vanishing transition states. In Sec. IV, we focus on a specific example of two Calabi-Yau spaces related by a flop and study the cosmological evolution numerically. We conclude in Sec. V.

II. THE FIVE-DIMENSIONAL EFFECTIVE ACTION OF M-THEORY

To set the notation, we will first review $N=1$ supergravity in eleven and five dimensions and the structure of the five-dimensional effective action for M-theory on Calabi-Yau threefolds. Subsequently, we will show how to couple to this action the hypermultiplet which contains the transition states. The five-dimensional effective action, including this hypermultiplet, will be the basis for the subsequent cosmological analysis.

A. Supergravity in eleven and five dimensions

The bosonic part of 11 dimensional supergravity is given by [13]

$$S_{11} = -\frac{1}{2\kappa^2} \int_{M_{11}} \left\{ d^{11}x \sqrt{-g} \left(\frac{1}{2} R + \frac{1}{4!} G_{IJKL} G^{IJKL} \right) + \frac{2}{3} C \wedge G \wedge G \right\}, \tag{2.1}$$

where $G = dC$ is the field strength of the three-form potential C and κ is the 11-dimensional Newton constant, as usual. Indices $I,J,K, \ldots = 0, \ldots, 10$ label the 11-dimensional coordinates x^I. Later, we will also need the bosonic part of the membrane action [14,15]

$$S_{M_3} = -T_2 \int_{M_3} \{ d^3\sigma \sqrt{-\gamma} + 2\hat{C} \} \tag{2.2}$$

which couples to C. The membrane world volume is parametrized by coordinates σ^n, where $n,m,p, \ldots = 0,1,2$, and its embedding into 11-dimensional space-time is specified by $X^I = X^I(\sigma)$. The pull-backs γ_{nm} and $\hat{C}$ of the space-time metric and the three-form are defined by

$$\gamma_{nm} = \partial_n X^I \partial_m X^J g_{IJ}, \tag{2.3}$$

$$\hat{C}_{nmp} = \partial_n X^I \partial_m X^J \partial_p X^K C_{IJK}, \tag{2.4}$$

as usual. In terms of the 11-dimensional Newton constant, the membrane tension T_2 is given by

$$T_2 = \left(\frac{8\pi}{\kappa^2} \right)^{1/3}. \tag{2.5}$$

Let us now move on to five-dimensional $N=1$ supergravity focusing on the aspects relevant to this paper. For a more complete account we refer to the literature [8–12].

We denote five-dimensional space-time indices by $\alpha, \beta, \gamma, \ldots = 0,1,2,3,4$. In addition to the supergravity multiplet, consisting of the vielbein, an Abelian vector field and the gravitini, there are two types of matter multiplets, namely vector- and hyper-multiplets. In general, one can have any number, n_V, of vector multiplets, each containing a real scalar field and an Abelian vector field plus fermionic partners and any number, n_H, of hypermultiplets, each containing four real scalars plus fermions. It is useful to treat the Abelian gauge fields in the vector multiplets and the supergravity multiplet on the same footing and collectively denote them by A_α^i, where $i,j,k, \ldots = 0, \ldots, n_V$. The real scalars contained in the vector multiplets are described by $n_V + 1$ fields b^i. These define a manifold of very special geometry [9] with metric

$$G_{ij} = -\partial_i \partial_j \ln K, \tag{2.6}$$

which is given in terms of the degree three homogeneous polynomial

$$K = d_{ijk} b^i b^j b^k. \tag{2.7}$$

Here d_{ijk} are constant coefficients. The b^i are subject to the constraint

$$K = 6 \tag{2.8}$$

which reduces the number of independent fields to n_V, as required.

Further, we denote by Q^u, where $u,v,w, \ldots = 1, \ldots, 4n_H$, the hypermultiplet scalars. They parametrize a quaternionic manifold, that is, a manifold with holonomy $SU(2) \times Sp(2n_H)$. The metric on this manifold, h_{uv}, is Hermitian with respect to the three complex structures

$$J_u^{\ v} = J_u^{a\ v} \tau_a \tag{2.9}$$

satisfying the quaternionic algebra

$$J_u^{a\ v} J_v^{bw} = -\delta^{ab} \delta_u^{\ w} + \epsilon^{abc} J_u^{c\ w}, \tag{2.10}$$

where $a,b,c, \ldots = 1,2,3$. Here τ_a are the Hermitian Pauli matrices so that the complex structures fill out the adjoint of $SU(2)$. The associated triplet of Kähler forms is given by

$$\mathcal{K}_{uv} = J_u^{\ w} h_{wv}. \tag{2.11}$$

We also need to introduce the $SU(2)$ part $\omega_u = \omega_u^a \tau_a$ of the spin connection.

Let us assume that the metric h_{uv} admits $n_V + 1$ Killing vectors k_i^u. These Killing vectors should respect the quaternionic structure which means they originate from prepotentials $\mathcal{P}_i = \mathcal{P}_i^a \tau_a$ via the relation

$$k_i^u \mathcal{K}_{uv} = \partial_v \mathcal{P}_i + [\omega_v, \mathcal{P}_i]. \tag{2.12}$$

With these conventions the bosonic part of the supergravity and vector multiplet action reads

$$S_V = -\frac{1}{2\kappa_5^2} \int_{M_5} \left\{ d^5x \sqrt{-g} \left(\frac{1}{2}R + \frac{1}{4}G_{kl}\partial_\alpha b^k \partial^\alpha b^l \right. \right.$$
$$\left. \left. + \frac{1}{2}G_{kl}F^k_{\alpha\beta}F^{l\alpha\beta} \right) + \frac{2}{3}d_{klm}A^k \wedge F^l \wedge F^m \right\}, \quad (2.13)$$

where κ_5 is the five-dimensional Newton constant. The bosonic part of the hypermultiplet action takes the form

$$S_H = -\frac{1}{2\kappa_5^2} \int_{M_5} d^5x \sqrt{-g} \{ h_{uv}D_\alpha Q^u D^\alpha Q^v + V \}, \quad (2.14)$$

with the potential V given by

$$V = \frac{1}{2}g^2 \left[4(G^{ij} - b^i b^j)\mathrm{tr}(\mathcal{P}_i \mathcal{P}_j) + \frac{1}{2}b^i b^j h_{uv}k_i^u k_j^v \right]. \quad (2.15)$$

The trace in this expression is performed over the Pauli matrices. The covariant derivative D_α includes the gauging of the hypermultiplets with respect to the vector fields A^i and is defined by

$$D_\alpha Q^u = \partial_\alpha Q^u + gA^i_\alpha k_i^u, \quad (2.16)$$

where g is the gauge coupling. Note that the appearance and the structure of the potential (2.15) is directly linked to this gauging of the hypermultiplets.

B. M-theory on Calabi-Yau threefolds

Let us now briefly review the reduction [16] of the action (2.1) for 11-dimensional supergravity on a Calabi-Yau threefold X with Hodge numbers $h^{1,1}$ and $h^{2,1}$. This leads to a five-dimensional $N=1$ supergravity theory of the type described in the previous section with $n_V = h^{1,1} - 1$ vector multiplets, $n_H = h^{2,1} + 1$ hypermultiplets and no gauging of the hypermultiplets.

We now need to specify the geometrical origin of some of these five-dimensional fields. The 11-dimensional metric on the direct product space $M_{11} = M_5 \times X$ can be written as

$$ds_{11}^2 = \mathcal{V}^{-2/3}g_{\alpha\beta}dx^\alpha dx^\beta + g_{AB}dx^A dx^B, \quad (2.17)$$

where $A,B,C,\ldots = 5,\ldots,10$ label the coordinates on the Calabi-Yau space, g_{AB} is the Ricci-flat metric on X and $g_{\alpha\beta}$ is the five-dimensional metric. The Calabi-Yau volume modulus $\mathcal{V}$ is defined by

$$\mathcal{V} = \frac{1}{v} \int_X d^6x \sqrt{g_6}, \quad (2.18)$$

where v is an arbitrary six-dimensional reference volume. We note that the five-dimensional Newton constant κ_5 is related to its 11-dimensional counterpart κ by

$$\kappa_5^2 = \frac{\kappa^2}{v}. \quad (2.19)$$

The Kähler form ω_{AB} of X can be expanded as

$$\omega_{AB} = a^i \omega_{iAB} \quad (2.20)$$

into a basis $\{\omega_{iAB}\}$ of $(1,1)$ forms. We take this basis of $(1,1)$ forms to be dual to an (effective) basis $\{\mathcal{W}^i\}$ of the second homology, that is,

$$v^{-1/3} \int_{\mathcal{W}^i} \omega_j = \delta_j^i. \quad (2.21)$$

The $h^{1,1}$ expansion coefficients a^i are the Kähler moduli of the Calabi-Yau space. As stands, they are, of course, not independent from the volume modulus $\mathcal{V}$. However, we can define volume-independent moduli b^i by

$$b^i = \mathcal{V}^{-1/3}a^i. \quad (2.22)$$

These constitute $n_V = h^{1,1} - 1$ independent fields as they can be shown to satisfy the constraint (2.8). They should be interpreted as the scalar fields in the vector multiplets. The coefficients d_{ijk} which appear in Eq. (2.7) should then be identified as the intersection numbers of the Calabi-Yau space.

The three-form C can be expanded as

$$C = \bar{C} + A^i \wedge \omega_i + \xi \wedge \Omega + \bar{\xi} \wedge \bar{\Omega}, \quad (2.23)$$

where Ω is the holomorphic $(3,0)$ form on X. The $h^{1,1}$ five-dimensional vector fields A^i_α account for the gauge fields in the vector multiplets and in the gravity multiplet. The five-dimensional three-form $\bar{C}$ can be dualized to a scalar and forms, together with the complex scalar ξ and the volume modulus $\mathcal{V}$, the universal hypermultiplet. There are $h^{2,1}$ additional hypermultiplets which originate from the complex structure moduli and the $(2,1)$ part of C which we have omitted in Eq. (2.23). These standard hypermultiplets will not be of particular importance in the following.

Let us now review some relevant features of M-theory flop transitions following Refs. [2,6]. A flop constitutes a transition from the Calabi-Yau space X to a topologically different space $\tilde{X}$ due to a complex curve $\mathcal{C}$ in X shrinking to zero size and subsequently being enlarged in a topologically distinct way. Concretely, let us expand the class $\mathcal{W}$ of the curve $\mathcal{C}$ in our homology basis as

$$\mathcal{W} = \beta_i \mathcal{W}^i \quad (2.24)$$

with constant coefficients β^i. The volume of $\mathcal{C}$ can then be written as

$$\mathrm{Vol}(\mathcal{C}) = \int_{\mathcal{C}} \omega = (v\mathcal{V})^{1/3}b \quad (2.25)$$

where we have introduced the particular linear combination

$$b = \beta_i b^i \quad (2.26)$$

of vector multiplet moduli. Within the moduli space of X we have $b^i > 0$, for all i, as well as $b > 0$ and the limit $b \to 0$ corresponds to approaching the flop. Continuing further to

negative values of b leads into the moduli space of the birationally equivalent Calabi-Yau space $\widetilde{X}$. This new Calabi-Yau space $\widetilde{X}$ has the same Hodge numbers and, hence, the same five-dimensional low-energy spectrum as the original space X. However, the intersection numbers have changed across the transition [2]. More specifically, setting $(\beta_i)=(1,0,\ldots,0)$ for simplicity the new intersection numbers $\widetilde{d}_{ijk}$ (expressed in terms of the field basis b^i) are given by

$$\widetilde{d}_{111}=d_{111}-\frac{1}{6} \tag{2.27}$$

with all other components unchanged. Sometimes a new basis of fields $\widetilde{b}^i$, defined by

$$\widetilde{b}^1=-b^1, \quad \widetilde{b}^i=b^i-b^1, \tag{2.28}$$

for all $i\neq 1$, is introduced [6] to cover the moduli space of $\widetilde{X}$. These new fields have the advantage of being positive throughout the moduli space of $\widetilde{X}$ which is not the case for the original fields b^i. For our applications we will find it usually more practical to use a single set of fields to cover the moduli spaces for both X and $\widetilde{X}$.

How does the five-dimensional effective theory change across the transition? Inspection of the action (2.13), (2.14) without gauging and potential shows that only the vector multiplet part (2.13) is affected through the change (2.27) in the intersection numbers. From Eq. (2.6), the metric G_{ij} takes the specific form

$$G_{ij}=-d_{ijk}b^k+\frac{1}{4}d_{ikl}d_{jmn}b^k b^l b^m b^n, \tag{2.29}$$

where we have used that $K=6$. This form shows that, despite the jump (2.27) in the intersection number the metric remains continuous across the flop since d_{111} is always multiplied by b^1 which vanishes at the transition. We remark that the associated connection

$$\Gamma_{ij}^k=\frac{1}{2}G^{kl}\frac{\partial G_{ij}}{\partial b^l} \tag{2.30}$$

which appears in the five-dimensional equations of motion contains a term proportional to d_{ijk} (without additional fields b^i) and, hence, jumps across the flop. Given the continuity of the metric G_{ij} the only discontinuous term in the action is the Chern-Simons term in Eq. (2.13) which is proportional to the intersection numbers. It has been shown [2] that its jump can be accounted for by loop corrections which involve the transition states. Let us now discuss these additional states in more detail.

C. The transition states

The five-dimensional particles which become massless at the flop originate from a membrane which wraps the collapsing complex curve $\mathcal{C}$ with homology class $\mathcal{W}$. We can find the world-line action for these transition states by starting with the membrane action (2.2). Introducing a complex world-volume coordinate $\sigma=\sigma^1+i\sigma^2$ and world time $\tau=\sigma^0$ we consider an embedding of the membrane into 11-dimensional space of the form

$$X^\alpha=X^\alpha(\tau), \quad X^A=X^A(\sigma), \quad X^{\bar{A}}=X^{\bar{A}}(\bar{\sigma}), \tag{2.31}$$

where here A and $\bar{A}$ are holomorphic and anti-holomorphic indices on the Calabi-Yau space, respectively, and $X^A=X^A(\sigma)$ parametrizes the complex curve $\mathcal{C}$. The reduction of the membrane action on this curve leads to the following world-line action:

$$S_p=-(v^{1/3}T_2)\int_{\mathbb{R}}\{dt(\beta_i b^i)\sqrt{-\partial_\tau X^\alpha\partial_\tau X^\beta g_{\alpha\beta}}+2\beta_i\hat{A}^i\}. \tag{2.32}$$

This particle must form a hypermultiplet in five dimensions. We denote the scalars in this hypermultiplet by q^u, where $u,v,w,\ldots=1,2,3,4$. It is charged with respect to the particular linear combination

$$\mathcal{A}\equiv\beta_i A^i \tag{2.33}$$

of vector fields with associated gauge coupling

$$g=2v^{1/3}T_2=2\left(\frac{8\pi}{\kappa_5^2}\right)^{1/3}, \tag{2.34}$$

as can be seen from the last term in Eq. (2.32). From the first term in the world-line action we can read off the mass which is given by

$$m=\frac{1}{2}gb=T_2\mathcal{V}^{-1/3}\text{Vol}(\mathcal{C}). \tag{2.35}$$

What does this information tell us about the five-dimensional effective action of these transition states? Clearly, these states being hypermultiplets, their effective action must be of the general form (2.14). We assume that the associated hypermultiplet moduli space metric is flat, so that

$$h_{uv}=\delta_{uv}. \tag{2.36}$$

As we will see shortly, this assumption is consistent with the above properties of the transition states and the constraints enforced by five-dimensional supergravity. To work this out explicitly, let us first recall the quaternionic structure on the four-dimensional flat moduli space. Introducing the 't Hooft η symbols [17]

$$\eta_{bc}^a=\bar{\eta}_{bc}^a=\epsilon_{bc}^a, \tag{2.37}$$

$$\eta_{b0}^a=\bar{\eta}_{0b}^a=\delta_b^a, \tag{2.38}$$

which satisfy the properties

$$[\eta^i,\bar{\eta}^j]=0,$$

$$(\eta^i)^T=(\eta^i)^{-1}, \quad (\bar{\eta}^i)^T=(\bar{\eta}^i)^{-1}, \tag{2.39}$$

the triplet of complex structures can be written as

$$J^a{}_u{}^v \equiv -\bar{\eta}^a_{uw}\delta^{wv}, \qquad (2.40)$$

which satisfy the quaternionic algebra (2.10), as required. The associated triplet of Kähler forms is given by

$$\mathcal{K}^a_{uv} = -\bar{\eta}^a_{uv}. \qquad (2.41)$$

We know that the transition states are charged under the particular combination of gauge fields (2.33). Hence the vectors k^u_i must be proportional to β_i and, at the same time, be Killing vectors on flat four-dimensional space. We know that this gauging must lead to a potential of the form (2.15). As $q^u \rightarrow 0$ this potential must vanish so that the moduli b^i indeed parametrize flat directions in this limit. This implies that the Killing vectors k^u_i should not correspond to translations but rather to rotations and, hence, be of the form

$$k^u_i = \beta_i t^u{}_v q^v, \qquad (2.42)$$

where t is an arbitrary anti-symmetric matrix. In addition, these Killing vectors must originate from a prepotential, that is, they must satisfy Eq. (2.12). This is the case precisely if $[t,\bar{\eta}^a]=0$ for $a=1,2,3$, or, equivalently, if the matrix t is of the form

$$t_{uv} = n_a \eta^a_{uv}, \qquad (2.43)$$

where n_a are real coefficients. This matrix represents the generator of $SO(2)$ in the representation $\mathbf{2}\oplus\mathbf{2}$. We require the standard normalization $\mathrm{tr}(t^2)=-4$ or, equivalently, $n_a n^a = 1$. The associated prepotential then reads

$$\mathcal{P}^a_i = \frac{1}{2}\beta_i q^v (\bar{\eta}^a_{vw} n_b \eta^{bw}{}_u) q^u + \xi^a_i, \qquad (2.44)$$

where ξ^a_i are arbitrary integration constants. They represent the generalization of Fayet-Illiopoulos terms to five-dimensional $N=1$ supergravity. As they lead to terms in the potential which do not vanish for vanishing q^u we will set them to zero in the following.

Inserting Eqs. (2.36), (2.42), (2.43) and (2.44) into the general hypermultiplet action (2.14) we obtain

$$S_q = -\frac{1}{2\kappa_5^2}\int_{M_5}d^5x\sqrt{-g}\{D_\alpha q^u D^\alpha q_u + V\}, \quad (2.45)$$

with the potential

$$V = \frac{1}{4}g^2[b^2 q_u q^u + 4(G^{kl}\beta_k\beta_l - b^2)(q_u q^u)^2] \quad (2.46)$$

and the covariant derivative

$$D_\alpha q^u = \partial_\alpha q^u + g\mathcal{A}_\alpha t^u{}_v q^v. \qquad (2.47)$$

Consequently, the hypermultiplet current j_α which couples to the gauge field $\mathcal{A}_\alpha$ is given by

$$j_\alpha = g q^u t_{uv}\partial_\alpha q^v. \qquad (2.48)$$

We recall that $b=\beta_i b^i$, defined in Eq. (2.26), is proportional to the volume of the collapsing cycle and the generator t has been given in Eq. (2.43). So far, we have only used that the transition states are charged under the gauge field $\mathcal{A}$. Clearly, the gauge coupling g which appears in the covariant derivative (2.47) has to be identified with the value (2.34) obtained from the reduction of the membrane action. Then, the above hypermultiplet action (2.45) is completely fixed. From the first term in the potential (2.46) we can now read off the mass of the hypermultiplet which is given by $gb/2$. This value indeed coincides with the one obtained from the membrane reduction, Eq. (2.35), as it should for consistency.

In addition, we have found a potential term quartic in the transition states which was not anticipated from the membrane reduction but imposed on us by five-dimensional supergravity. This quartic term plays an important role in lifting "unwanted" flat directions. While the potential should be flat for vanishing transition states, $q^u=0$, and arbitrary b^i, a flat direction along the flop at $b=0$ and arbitrary q^u would be a surprise.[1] Fortunately, this potential flat direction is lifted by the second term in Eq. (2.46).

III. COSMOLOGY

A. Cosmological ansatz and equations of motion

Let us briefly summarize the discussion so far. We have seen that M-theory on a Calabi-Yau threefold X with Hodge numbers $h^{1,1}$ and $h^{2,1}$ is effectively described by the five-dimensional supergravity action (2.13), (2.14) with $n_V = h^{1,1}-1$ vector multiplets and $n_H = h^{2,1}+1$ hypermultiplets and no gauging. When a flop transition to a topologically distinct Calabi-Yau space $\tilde{X}$ occurs the Hodge numbers and hence the number of massless particles remains the same while the structure of the five-dimensional action changes in accordance with the change (2.27) in the intersection numbers. In addition, at and near the flop-transition region another light hypermultiplet appears whose action (2.45) has to be added to the previous one for an accurate description across the transition. It is important not to confuse these transition hypermultiplet states which arises at the flop with the standard hypermultiplets associated with the complex structure moduli space of the Calabi-Yau space.

Which parts of this five-dimensional effective action are we actually interested in for our cosmological applications? Since we would like to study flop transitions which arise by moving in the Calabi-Yau Kähler moduli space we should certainly consider the associated moduli fields, that is, the

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[1]Such a flat direction with non-vanishing transition states would correspond to a Higgs branch where the gauge symmetry corresponding to the vector field $\mathcal{A}$ is broken. The resulting change in the number of light vector multiplets would be inconsistent with the fact that Hodge numbers are unchanged across the flop.

vector multiplet scalars b^i. Clearly, we should also keep the transition states q^u which become light at the flop. However, these states are charged and generically source the vector fields. Hence, it seems we have to allow for non-trivial vector field backgrounds for consistency. Fortunately, we can avoid such a considerable complication by setting all scalars q^u equal to each other, that is, $q \equiv 2q^u$ for all $u=1,2,3,4$ and a single scalar q. This configuration is consistent with the q^u equations of motion, as can be seen from Eq. (2.46), and leads to a vanishing current (2.48). Consequently, the vector fields can be consistently set to zero in this case. The standard hypermultiplets, in fact, completely decouple from the other fields and are, hence, not essential for our purpose. From these standard hypermultiplet scalars we will only keep the dilaton $\mathcal{V}=e^\phi$ since it represents the overall volume of the internal Calabi-Yau space and is, therefore, of particular physical relevance.

In summary, the spectrum of our five-dimensional effective action can be consistently truncated to the five-dimensional metric, the $h^{1,1}-1$ Kähler moduli space scalars b^i, the universal transition scalar q and the dilaton ϕ. From Eq. (2.13) and Eq. (2.45), the accordingly truncated effective action then reads

$$S_5 = -\frac{1}{2\kappa_5^2}\int_{M_5}d^5x\sqrt{-g}\left\{\frac{1}{2}R + \frac{1}{4}\partial_\alpha\phi\partial^\alpha\phi\right.$$
$$\left. + \frac{1}{4}G_{kl}\partial_\alpha b^k\partial^\alpha b^l + \lambda(K-6) + \partial_\alpha q\partial^\alpha q + V\right\},$$
$$(3.1)$$

$$V = \frac{1}{4}g^2[(\beta_l b^l)^2q^2 + 4(G^{kl}\beta_k\beta_l - (\beta_l b^l)^2)q^4]. \quad (3.2)$$

A Lagrange multiplier term has been added to enforce the constraint $K=6$, Eq. (2.8), on the moduli b^i. The value of the gauge coupling g has been given in Eq. (2.34). We also recall that the "Kähler potential" K and the metric G_{ij} have been defined in Eq. (2.7) and Eq. (2.6), respectively.

We are now ready to consider the cosmological evolution of our system. We focus on backgrounds depending on time τ only and a metric with a three-dimensional maximally symmetric subspace which we take to be flat, for simplicity. Accordingly, we consider the following Ansatz:

$$ds^2 = -e^{2\nu(\tau)}d\tau^2 + e^{2\alpha(\tau)}d\mathbf{x}^2 + e^{2\beta(\tau)}dy^2 \quad (3.3)$$

$$\phi = \phi(\tau) \quad (3.4)$$

$$b^i = b^i(\tau) \quad (3.5)$$

$$q = q(\tau), \quad (3.6)$$

where $\mathbf{x} = (x^1, x^2, x^3)$ and $y = x^4$. Note that α and β are the scale factors of the three-dimensional universe and the additional spatial dimension, respectively. For later convenience, we have also included a lapse function ν. The equations of motion for this Ansatz, derived from the action (3.1), are given by Einstein equations

$$3(\dot{\alpha}^2 + \dot{\alpha}\dot{\beta})$$
$$= +\frac{1}{4}(\dot{\phi}^2 + G_{ij}\dot{b}^i\dot{b}^j + 4\dot{q}^2) + e^{2\nu}V$$

$$3(\ddot{\alpha} - \dot{\nu}\dot{\alpha} + 2\dot{\alpha}^2)$$
$$= -\frac{1}{4}(\dot{\phi}^2 + G_{ij}\dot{b}^i\dot{b}^j + 4\dot{q}^2) + e^{2\nu}V$$

$$2\ddot{\alpha} + \ddot{\beta} + 3\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta} - 2\dot{\nu}\dot{\alpha} - \dot{\nu}\dot{\beta}$$
$$= -\frac{1}{4}(\dot{\phi}^2 + G_{ij}\dot{b}^i\dot{b}^j + 4\dot{q}^2) + e^{2\nu}V \quad (3.7)$$

and field equations of motion

$$\ddot{\phi} + (3\dot{\alpha} + \dot{\beta} - \dot{\nu})\dot{\phi} = 0 \quad (3.8)$$

$$\ddot{b}^k + (3\dot{\alpha} + \dot{\beta} - \dot{\nu})\dot{b}^k + \Gamma^k_{ij}\dot{b}^i\dot{b}^j$$
$$+ 2e^{2\nu}(G^{kj}\partial V/\partial b^j - \tfrac{2}{3}b^kV) = 0$$

$$\ddot{q} + (3\dot{\alpha} + \dot{\beta} - \dot{\nu})\dot{q} + \frac{1}{2}e^{2\nu}\frac{\partial V}{\partial q} = 0 \quad (3.9)$$

$$K = 6.$$

In these equations, we have already used the result $\lambda = -V/9$ for the Lagrange multiplier λ which follows by contracting the b^i equations of motion[2] with $b_i = G_{ij}b^j$. The connection Γ^k_{ij} associated with the moduli space metric G_{ij} has been defined in Eq. (2.30).

The above action and evolution equations have been written for a definite topology of the internal space with intersection numbers d_{ijk}. When an evolution leads to a flop transition, the Kähler potential K, the metric G_{ij} and the connection Γ^k_{ij} have to be changed "by hand" in accordance with the change (2.27) in the intersection numbers to obtain the equations of motion for the new topology. As discussed earlier, this implies continuity of K and the metric while the connection jumps across the transition. We also note that, from Eq. (3.2), the potential V is continuous while its derivatives with respect to b^i contain the connection and, hence, jump. From these properties and the equations of motion we conclude that all fields and their first time derivatives and, hence, the stress energy for all fields is continuous across the flop.

B. An approximate solution for vanishing transition states

It is clear from the Eqs. (3.7) and (3.9) that the transition state q can be set to zero consistently and we will now analyze the cosmological evolution in this case. From the structure of the equations of motion, the configuration $q=0$

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[2]Some relations for very special geometry which are useful in this context have been collected in Ref. [11].

seems rather non-generic and having a non-vanishing, perhaps small initial value for q appears to be more plausible. We will study this generic case further below. However, setting $q=0$ corresponds to the conventional picture of a flop transition as being induced by slow free rolling in moduli space. It is, therefore, useful to consider this case in some detail, if only as a point of reference.

Setting the transition state q and, hence, the potential V, to zero simplifies the equations of motion considerably. Another simplification arises if we choose the gauge $\nu = 3\alpha + \beta$ for the lapse function. In this gauge, we will denote time by τ in the following. The second term in the b^i equations vanishes for this choice and multiplying the remainder by $\dot{b}_i$ we find by integration that

$$k \equiv G_{ij} \frac{\partial b^i}{\partial \tau} \frac{\partial b^j}{\partial \tau} = \text{const}. \qquad (3.10)$$

Hence, the kinetic energy k of the Kähler moduli is constant on hypersurfaces of constant time τ. Note that this is no longer true for proper cosmological time related to τ by $dt^2 = e^{6\alpha + 2\beta} d\tau^2$. It is straightforward to integrate the Einstein equations (3.7) and the ϕ equation of motion for time τ. The result can be easily rewritten in terms of proper time t where it takes the form

$$\alpha = p_\alpha \ln|t|, \quad \beta = p_\beta \ln|t|, \quad \phi = p_\phi \ln|t|. \qquad (3.11)$$

Here, we have dropped trivial additive integration constants for all three fields and the origin of time, for simplicity. The expansion powers p_α, p_β and p_ϕ must satisfy the constraints

$$3p_\alpha + p_\beta = 1, \qquad (3.12)$$

$$p_\alpha^2 + p_\alpha p_\beta = \frac{1}{12}(p_\phi^2 + k), \qquad (3.13)$$

and the relation between proper time t and τ is simply

$$\tau = \ln|t|. \qquad (3.14)$$

We still need to find the explicit form of b^i for a complete solution. To do this, we have to solve the following system of equations:

$$\ddot{b}^i + \Gamma^i_{jk} \dot{b}^j \dot{b}^k = 0 \qquad (3.15)$$

$$d_{ijk} b^i b^j b^k = 6 \qquad (3.16)$$

$$G_{ij} \dot{b}^i \dot{b}^j = k. \qquad (3.17)$$

Here the dot denotes the derivative with respect to τ. Hence, in this time coordinate, the fields b^i move along geodesics in moduli space subject to the constraint (3.16) from special geometry and the kinetic energy constraint (3.17). Unfortunately, Eq. (3.15) is hard to solve in general due to the second non-linear term. However, for a sufficiently small time interval and slow motion this term can be neglected. In other words, the geodesics are well approximated by straight lines

$$b^i = c^i + p^i \tau + O(\tau^2) \qquad (3.18)$$

in moduli space, where c^i and p^i are constants, as long as

$$|\tau| \ll \left| \frac{2p^i}{\Gamma^i_{jk}(c) p^j p^k} \right|. \qquad (3.19)$$

This approximation also implies that we neglect the kinetic energy of the fields b^i compared to the other fields, that is, from Eq. (3.17), we consider a solution with $k \simeq 0$. Accordingly, this value for k has to be inserted into the relation (3.13). Within our approximation, the special geometry constraint (3.16) turns into two conditions, namely,

$$d_{ijk} c^i c^j c^k = 6, \quad d_{ijk} c^i c^j p^k = 0. \qquad (3.20)$$

These algebraic equations can be easily solved for given intersection numbers. This completes our approximate solution.

Let us now apply this result to a flop transition. We assume, for simplicity, that the flop occurs in the b^1 direction and at $b^1 = 0$. By setting $c^1 = 0$ in our solution (3.18) we can, in fact, arrange the flop to take place at time $\tau = 0$. Now we consider two solutions of the above type with intersection numbers d_{ijk} and $\tilde{d}_{ijk}$. We recall that d_{111} is the only intersection number which changes [as given in Eq. (2.27)] across the transition. Since we have set $c^1 = 0$ this particular intersection number drops out of the constraints (3.20) which need to be satisfied for a valid solution. We are, therefore, free to choose the same constants c^i, p^i, solving the constraints (3.18), on both sides of the flop. This leads to two solutions, for either topology, which can be continuously matched together at the flop transition for $\tau = 0$. Further, our approximation is valid for a certain period of time before and after the flop as quantified by the condition (3.19). The scale factors α and β and the dilaton ϕ are unaffected by the transition in that they evolve according to Eq. (3.11) with the same expansion powers p_α, p_β and p_ϕ on both sides of the transition. These results suggest that the system indeed evolves through the transition into the moduli space of the topologically distinct Calabi-Yau space and, hence, that the topology change is dynamically realized. This picture will be confirmed by our numerical integration further below.

C. Evolution for non-vanishing transition states

If the transition states no longer vanish, that is $q \neq 0$, the potential becomes operative and the conclusions of the previous section do not apply any more. Clearly, we should not expect to find analytic solutions in this case any more. However, some qualitative features of the evolution can be inferred from the structure of the potential (3.2).

Let us consider small fixed values of the transition state such that the potential (3.2) is dominated by the first term. It is then approximately given by $V \sim b^2 q^2$. Note that this potential for fixed non-zero q has a minimum at $b = 0$, that is, precisely at the flop point. From this observation and the general shape of the potential it is intuitively clear that a generic evolution will lead to oscillations around $b = 0$ and

will finally settle down to this point. In other words, there is a clear preference for the system to settle down at the transition point rather than complete the transition.

The complete potential (3.2) must still have a minimum at small b for fixed non-zero q at least as long as q is sufficiently small. This suggests that the above argument generalizes to this case and this will indeed be confirmed by our numerical integration. In conclusion, this suggest that the system behaves quite differently if we allow non-zero values of the transition states. Previously, for vanishing transition states, we have found that the topology does change dynamically. If $q \neq 0$, on the other hand, the potential becomes important and favors the region in moduli space close to the flop transition. In this case, the system tends to settle down in the transition region so that the topology change is not completed. It should be noted though that since the potential (3.2) is only effective as long as the transition states q are non-vanishing, for an evolution with $q \rightarrow 0$ the flop might still be completed after a period of initial oscillations around the flop point. Moreover, the given conclusions are based on the analysis of the purely classical system considered, not taking any other influences into account [18], such as for example, a non-perturbative potential from gaugino condensation [19].

IV. AN EXPLICIT MODEL

In this section we would like to substantiate our previous claims by numerically studying the cosmological evolution of our system. To do this we need to consider a particular example, that is, a particular pair of Calabi-Yau manifolds related by a flop transition which provides us with a concrete set of intersection numbers. We will use the Calabi-Yau spaces described in Refs. [20–22] and applied to black hole physics in Refs. [6,7].

Concretely, we consider two elliptically fibred Calabi-Yau spaces X and $\tilde{X}$, both with a Hirzebruch F_1 base space and with $h^{1,1} = 3$. These spaces share a boundary of the Kähler moduli space which corresponds to a flop transition. Following Ref. [6], both moduli spaces can be covered by a single set of coordinates (W, U, T) with the flop transition along $W = U$. The Kähler moduli space of X corresponds to the coordinate range

$$U > W > 0, \quad T > \frac{3}{2} U, \qquad (4.1)$$

and the associated Kähler potential is given by

$$K = \frac{9}{4} U^3 + 3 T^2 U - W^3. \qquad (4.2)$$

We can use the constraint $K = 6$ to solve for T in terms of the other two moduli resulting in

$$T = \frac{1}{6} \left(-\frac{27 U^3 - 12 W^3 - 72}{U} \right)^{1/2}. \qquad (4.3)$$

The fields b^i which we have previously used are related to (W, U, T) by

$$b^1 = U - W, \quad b^2 = W, \quad b^3 = T - \frac{3}{2} U. \qquad (4.4)$$

This definition implies, from Eq. (4.1), that the fields b^i are indeed positive throughout the moduli space of X. The flop transition is approached as $b^1 \rightarrow 0$. Hence, the coefficients (β^i) which enter the potential (3.2) are given by $(\beta^1, \beta^2, \beta^3) = (1, 0, 0)$.

For the second Calabi-Yau space $\tilde{X}$ the moduli are in the range

$$W > U > 0, \quad T > W + \frac{1}{2} U, \qquad (4.5)$$

and the Kähler potential is given by

$$\tilde{K} = \frac{5}{4} U^3 + 3 U^2 W - 3 U W^2 + 3 T^2 U. \qquad (4.6)$$

Solving $\tilde{K} = 6$ for T, as before, leads to

$$\tilde{T} = \frac{1}{6} \left(-\frac{15 U^3 + 36 U^2 W - 36 U W^2 - 72}{U} \right)^{1/2}. \qquad (4.7)$$

From Eq. (2.28), fields $\tilde{b}^i$ which are positive in the moduli space of $\tilde{X}$ can be defined by

$$\tilde{b}^1 = -b^1 = W - U, \quad \tilde{b}^2 = b^2 - b^1 = 2 W - U,$$

$$\tilde{b}^3 = b^3 - b^1 = T - \frac{5}{2} U + W. \qquad (4.8)$$

Note that the two moduli spaces (4.1) and (4.5) indeed have a common boundary at $b^1 = U - W = 0$ which is where the flop transition occurs. Also, using the basis b^i as defined above, it is easy to see that the Kähler potentials (4.2) and (4.6) are related by the shift (2.27) in the intersection numbers, as is required for a flop transition.

We will now study the above example using W and U or, equivalently, b^1 and b^2 as the independent variables. It is useful to plot the potential (3.2) as a function of these variables for a fixed value of q. This has been done in Fig. 1 for a value of $q = 1/3$. Obviously, this potential has a minimum in both directions which happens to be at

$$U = W = \left(\frac{3}{10} \right)^{1/3}, \qquad (4.9)$$

independent of the value of q. The associated potential value at the minimum (in units where $g = 1$) is

$$V_{\min} = \frac{300^{2/3}}{16} q^4. \qquad (4.10)$$

From our previous general argument, we did anticipate a minimum in the direction $b^1 = U - W$. However, a minimum
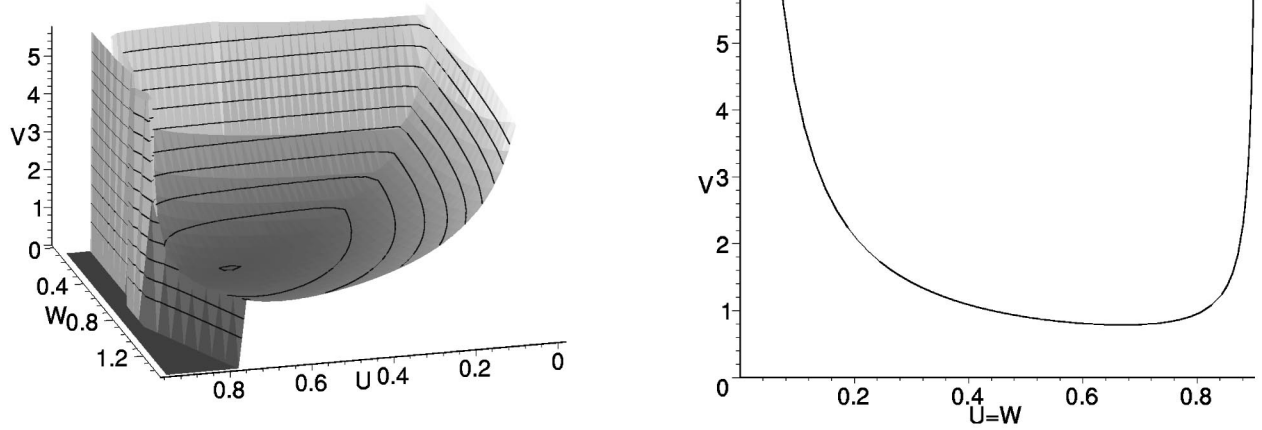
FIG. 1. Potential V in terms of W,U and along the flop transition line at W=U, both for fixed q=1/3.

for both fields, located precisely at the flop and, hence, at field values (4.9) independent of q, comes as a surprise. We do not know whether this is a general feature of the potential (3.2) near flop transitions or particular to this example. Incidentally, we note that, having fixed all moduli b^i, the shape of the potential (4.10) in the remaining q direction is well suited for inflation. Unfortunately, to be in the slow-roll regime we need that $q \gg 1$ which, in turn, implies that $V \gg 1$ in units of the fundamental Planck scale. Such potential values clearly go beyond the region of validity of our five-dimensional effective action and it is, therefore, difficult to conclude anything definite about this tantalizing possibility. In any case, we will restrict field values such that V does not become too large in the following.

Before studying the flop-transition numerically, let us briefly discuss the other boundaries of the moduli space as described by the conditions (4.1) and (4.5) in relation to the potential. We note that the potential steeply increases towards the boundary directions $W \rightarrow 0$, $W \rightarrow \infty$ and $U \rightarrow 0$. Hence, as long as the potential is operative (that is, q is non-zero) it prevents evolution towards these boundaries. The same is true for the boundary prescribed by $U^3 \rightarrow (W^3 + 6)/9$ as long as one stays in the X part of the moduli space, $b^1 = U - W > 0$, and W is sufficiently small. The potential barrier rapidly vanishes in this direction for increasing W in the $\widetilde{X}$ part of the moduli space. In our numerical evolution, we will simply avoid this direction of moduli space by choosing suitable initial conditions.

We have numerically integrated the system of equations (3.7), (3.9) for the above example, that is for the Kähler potentials (4.2) and (4.6). Here, we present the results for three characteristic sets of initial conditions which lead to an

evolution towards the flop transition region. The precise initial values of all fields are specified in Table I. Figure 2 shows the corresponding evolution of the fields as a function of proper time t. The first set of initial conditions in Table I leads to a vanishing transitions state, that is, we have chosen $q(0)=0$ and $\dot{q}(0)=0$. This is precisely the case we have discussed in Sec. III B. The resulting evolution is shown in the first column of Fig. 2. It can be seen that the system, starting off in the moduli space of X at $b^1 > 0$, evolves towards the flop transition $b^1 = U - W \rightarrow 0$ and then moves on to negative values of b^1, corresponding to the moduli space of $\widetilde{X}$. Hence, the topological transition is indeed dynamically realized as suggested by the previous analytic solution. The picture changes considerably once we allow for a non-vanishing transition state. The second set of initial conditions in Table I corresponds to small, non-vanishing values of $q(0)$ and $\dot{q}(0)$. Again, the system starts out in the X moduli space and evolves towards the flop. The associated plots in the second row of Fig. 2 show that after a few large initial oscillations around $b^1 = 0$ the system stabilizes at $b^1 = U - W \simeq 0$ and, hence, at the flop transition. A similar behavior can be observed for larger initial values $q(0)$, as in the third set in Table I with associated plots in the third column of Fig. 2.

V. CONCLUSIONS

We have shown in this paper that the dynamics of M-theory flop transitions strongly depends on whether or not the transition states which become light at the flop are taken into account. If these modes are exactly set to zero the

TABLE I. Table of initial conditions in order of increasing initial value of $q=q(0)$.

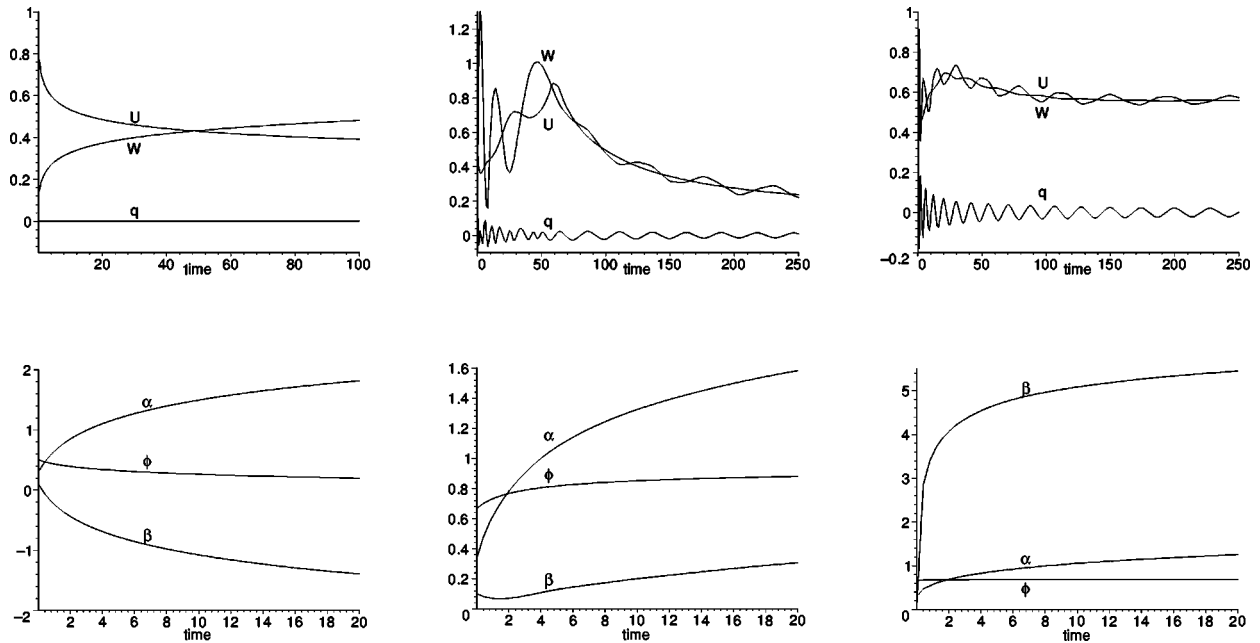| Plot	U	$\dot{U}$	W	$\dot{W}$	q	$\dot{q}$	ϕ	$\dot{\phi}$	α	$\dot{\alpha}$	β
1st	4/5	-1/5	1/8	1/9	0	0	1/2	-1/10	3/10	1/2	1/10
2nd	1/2	-1/5	1/5	1/10	1/5	1/8	2/3	1/10	1/3	2/5	1/10
3rd	4/5	-3/10	1/2	1/10	3/4	1/9	2/3	1/5	3/10	3/10	1/10

FIG. 2. Cosmological behavior of the fields $W(t)$, $U(t)$, $q(t)$ and $\alpha(t)$, $\beta(t)$, $\phi(t)$, respectively, for the three different sets of initial conditions given in Table I.

moduli space evolution proceeds freely and the topology change can indeed be dynamically realized, that is, the system moves between two topologically different Calabi-Yau spaces related by a flop for appropriate but generic initial conditions. For non-vanishing values of the transition modes, however, a potential becomes operative which generically leads to oscillations of moduli fields around the flop and thus to an approximate stabilization, at least as long as the potential is operative. Hence, the system does not really evolve into the moduli space of the flopped Calabi-Yau manifold and the transition remains incomplete. One may argue that this latter case is likely since non-vanishing values of the transition states represent a more generic set of field configurations in the early universe. If this is indeed the case, the region in moduli space close to a flop is preferred by the dynamics of the system.

It is likely that our results can be transferred to heterotic M-theory which provides a more realistic setting for low-energy physics from M-theory. To do this, we have to compactify the fifth dimension on a line interval and couple the five-dimensional $N=1$ bulk supergravity used in this paper to $N=1$ theories on the two boundaries [23,11]. The vacuum state of this theory is a static BPS domain wall which corresponds to a certain path in the Calabi-Yau Kähler moduli space as one moves between the boundaries. Using this property of the vacuum state, one can, in fact, construct static vacua of heterotic M-theory with an inherent flop transition [24], that is, vacua with the flop occurring at a particular point in the interval. Assuming the results of this paper indeed transfer to heterotic M-theory, it is these inherently flopped vacua which would be preferred by the dynamical evolution of the system.

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